

A further study of the possible scaling region of lattice chiral fermions

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In the possible scaling region for an $SU(2)$ lattice chiral fermion advocated in *Nucl. Phys. B* 486 (1997) 282, no hard spontaneous symmetry breaking occurs and doublers are gauge-invariantly decoupled via mixing with composite three-fermion-states that are formed by local multifermion interactions. However the strong coupling expansion breaks down due to no “static limit” for the low-energy limit ($pa \sim 0$). In both neutral and charged channels, we further analyze relevant truncated Green functions of three-fermion-operators by the strong coupling expansion and analytical continuation of these Green functions in the momentum space. It is shown that in the low-energy limit, these relevant truncated Green functions of three-fermion-states with the “wrong” chiralities positively vanish due to the generalized form factors (the wave-function renormalizations) of these composite three-fermion-states vanishing as $O((pa)^4)$ for $pa \sim 0$. This strongly implies that the composite three-fermion-states with “wrong” chirality are “decoupled” in this limit and the low-energy spectrum is chiral, as a consequence, chiral gauge symmetries can be exactly preserved.

I. INTRODUCTION

The attempt to get around the “no-go” theorem [1] for the “vector-like” phenomenon of chiral fermions on a lattice is currently a very important issue of theoretical particle physics ([2]–[13]). One attempt is the approach of multifermion couplings, which can be traced back from the recent lattice formulation of the Standard Model [14] to the pioneer model suggested by Eichten and Preskill in ref. [15] and successive work [16–18] in the past years. However, on the other hand, it was pointed out in an crucial paper [19] that the models proposed in ref. [15] fail to give chiral fermions in the continuum limit. The reasons are that a Nambu-Jona Lasinio (NJL) [20] spontaneous symmetry breaking phase separates the strong-coupling symmetric phase from the weak-coupling symmetric phase, and the right-handed Weyl states do not completely disassociate from the left-handed chiral fermions. Further, another crucial paper [21] tried to extent the “no-go” theorem into interacting theories based on the plausible argument of such theories being local. The definite failure of the models so constructed has then been a general belief [2] and it has been very doubtful that exact chiral gauge symmetries could be realized in a lattice. Nevertheless, based on the consistency of the Rome-approach [3] to an $SU(2)$ –chiral gauge theory on the lattice and physical nature of chiralities and the cutoff(lattice), we did strongly believe the existence of exact chiral gauge symmetries on the lattice and have been working on the multifermion couplings models for many years [16–18].

For reader’s convenience, we briefly describe the crucial points of the Eichten and Preskill [15] idea follows. Multifermion couplings are introduced such that, in the phase space of strong-couplings, Weyl states composing three elementary Weyl fermions (three-fermion states) are bound. Then, these three-fermion states pair up with elementary Weyl fermions to be Dirac fermions. Such Dirac fermions can be massive without violating chiral symmetries due to the appropriate quantum numbers and chirality carried by these three-fermion states. The binding thresholds of such three-fermion states depend on elementary Weyl modes residing in different regions of the Brillouin zone. If one assumes that the spontaneous symmetry breaking of the NJL-type does not occur and such binding thresholds separate the weak coupling symmetric phase from the strong-coupling symmetric phase, there are two possibilities to realize the continuum limit of chiral fermions in phase space. One is of crossing over the binding threshold the three-fermion state of chiral fermions. Another is of a wedge between two thresholds, where the three-fermion state of chiral fermions has not been formed, provided all doublers sitting in various edges of the Brillouin zone have been bound to be massive Dirac fermions and decouple.

We should not be surprised that a particular model of multifermion couplings does not work. This does not means that Eichten and Preskill’s idea is definitely wrong unless there is another generalized “no-go” theorem on interacting theories [21] for a whole range of coupling strength. Actually, Nielsen and Ninomiya gave an interesting comment on Eichten and Preskill’s idea based on their intuition of anomalies [26]. To conclude, we believe that further considerations of constructing chiral fermions on the lattice with multifermion couplings and careful studies of the spectrum in each phase of theories so constructed are necessary. In the previous paper [18] we illustrated a possible scaling regin for lattice chiral fermions in the phase space of multifermion couplings. In this paper, as a successive paper of [18] we go further to demonstrate the existence of the possible scaling regin for lattice chiral fermions by computing relevant Green functions and the analytical property of these Green functions in the momentum space.

The paper is organized as follow: section II summarizes the main points and results of the possible scaling region found in the previous paper [18]; in section III we discuss the momentum-dependent static limit and its analytical property in the momentum space, which is a crucial feature to have a possible scaling regin for lattice chiral fermions; in section IV we discuss the notion of three-fermion-cuts in the possible scaling regin; in section V we present very solid computations of relevant Green functions relating to the three-fermion-cuts for the neutral channel and conclude that neutral chiral fermions do exist in the possible scaling regin, whic already violates the “no-go” theorem; in section VI based on approximate computations of relevant Green-functions relating to the three-fermion-cuts for the charged channel and the analytical property of these Green-function in the momentum space, we show charged lattice chiral fermions could exist in such a scaling regin; in section VII we discuss the crucial properties of our model to work, what and why our model are different from the Eichten-Preskill and Smit-Swift model. In the final section, we make some remarks and outlook regarding this approach to lattice chiral gauge theories.

In this paper, we are not pretending to solve all problems of lattice chiral gauge theories in this paper. If chiral fermions are gauged, the important questions concerning correct and consistent features of gauge bosons and the coupling between gauge bosons and chiral fermions are open, but beyond the scope of this paper. Other important questions [4] concerning about gauge anomalies and anomalous global currents (instanton effects and non-conservation of fermion currents), which Eichten and Preskill [15] suggested to obtain by explicitly breaking the global symmetries associating to these currents, will be studied in separate papers.

II. A POSSIBLE SCALING REGION OF LATTICE CHIRAL FERMIONS

To more easily show the possibility and reason of dynamics for such constructed models to work, we further analyze the simple and anomaly-free model of multifermion couplings proposed in the ref. [17]. Note that ψ_L^i ($i = 1, 2$) is an $SU_L(2)$ gauged doublet, χ_R is an $SU_L(2)$ singlet and both are two-component Weyl fermions. χ_R is treated as a “spectator” fermion. ψ_L^i and χ_R fields are dimensionful $[a^{\frac{1}{2}}]$. The following action for chiral fermions with the $SU_L(2) \otimes U_R(1)$ chiral symmetries on the lattice is suggested:

$$\begin{aligned} S &= S_f + S_1 + S_2, \\ S_f &= \frac{1}{2a} \sum_x \sum_\mu \left(\bar{\psi}_L^i(x) \gamma_\mu D_{ij}^\mu \psi_L^j(x) + \bar{\chi}_R(x) \gamma_\mu \partial^\mu \chi_R(x) \right), \\ S_1 &= g_1 \sum_x \bar{\psi}_L^i(x) \cdot \chi_R(x) \bar{\chi}_R(x) \cdot \psi_L^i(x), \\ S_2 &= g_2 \sum_x \bar{\psi}_L^i(x) \cdot [\Delta \chi_R(x)] [\Delta \bar{\chi}_R(x)] \cdot \psi_L^i(x), \end{aligned} \quad (1)$$

where S_f is the naive lattice action for chiral fermions, a is the lattice spacing. S_1 and S_2 are two external multifermion couplings, where the g_1 and g_2 have dimension $[a^{-2}]$, and the Wilson factor is given as,

$$\begin{aligned} \Delta \chi_R(x) &\equiv \sum_\mu [\chi_R(x + \mu) + \chi_R(x - \mu) - 2\chi_R(x)], \\ 2w(p) &= \int_x e^{-ipx} \Delta(x) = \sum_\mu (1 - \cos(p_\mu)). \end{aligned} \quad (2)$$

Note that all momenta are scaled to be dimensionless, $p = \tilde{p} + \pi_A$ where π_A runs over fifteen lattice momenta ($\pi_A \neq 0$).

The action (1) has an exact local $SU_L(2)$ chiral gauge symmetry,

$$\sum_\mu \gamma_\mu D^\mu = \sum_\mu (U_\mu(x) \delta_{x, x+\mu} - U_\mu^\dagger(x) \delta_{x, x-\mu}), \quad U_\mu(x) \in SU_L(2), \quad (3)$$

which is the gauge symmetry that the continuum theory (the target theory) possesses. The global flavour symmetry $U_L(1) \otimes U_R(1)$ is not explicitly broken in eq. (1), we will not discuss the property of violating fermion number in such a model.

It has been advocated [17] there exists a plausible scaling region, which is a peculiar segment in the phase space of the multifermion couplings g_1, g_2 ,

$$\mathcal{A} = \left[g_1 \rightarrow 0, g_2^{c,a} < g_2 < g_2^{c,\infty} \right], \quad a^2 g_2^{c,a} = 0.124, \quad 1 \ll g_2^{c,\infty} < \infty, \quad (4)$$

for chiral fermions in the low-energy limit. $g_2^{c,\infty}$ is a finite number and $g_2^{c,a}$ indicates the critical value above which the effective multifermion couplings associating to all doublers are strong enough, so that all doublers are gauge-invariantly decoupled. We qualitatively determined $a^2 g_2^{c,a} = 0.124$ in ref. [17]. The crucial points for this scaling region to exist are briefly described in the following.

In segment \mathcal{A} (4), the action (1) possesses a χ_R -shift-symmetry [22], i.e., the action is invariant under the transformation:

$$\bar{\chi}_R(x) \rightarrow \bar{\chi}_R(x) + \bar{\epsilon}, \quad \chi_R(x) \rightarrow \chi_R(x) + \epsilon, \quad (5)$$

where ϵ is independent of space-time. The Ward identity corresponding to this χ_R -shift-symmetry is given as [17] ($g_1 \rightarrow 0$),

$$\frac{1}{2a} \gamma_\mu \partial^\mu \chi'_R(x) + g_2 \langle \Delta(\bar{\psi}_L^i(x) \cdot \Delta \chi_R(x) \psi_L^i(x)) \rangle - \frac{\delta \Gamma}{\delta \bar{\chi}_R(x)} = 0, \quad (6)$$

where the “primed” fields are defined through the generating functional approach, and “ Γ ” is the effective potential. The important consequences of this Ward identity in segment \mathcal{A} are:

- the low-energy mode ($p \sim 0$) of χ_R is a free mode and decoupled:

$$\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\delta \chi'_R(x) \delta \bar{\chi}'_R(0)} = \frac{i}{a} \gamma_\mu \sin(p^\mu); \quad (7)$$

- no hard spontaneous chiral symmetry breaking ($O(\frac{1}{a})$) occurs (see eqs.(30) and (31) in ref. [17])¹,

$$\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\delta \psi_L^i(x) \delta \bar{\chi}'_R(0)} = \frac{1}{2} \Sigma^i(p) = 0 \quad p = 0, \quad (8)$$

in addition, we have:

$$\Sigma(p) = 0 \quad p \neq 0 \quad (9)$$

which is shown by the strong coupling expansion (see eq.(104) in [17]).

For the strong coupling $g_2 \gg 1$ in the segment \mathcal{A} , the following three-fermion-states comprising of the elementary fields ψ_L^i and χ_R in (1) are bound:

$$\Psi_R^i = \frac{1}{2a} (\bar{\chi}_R \cdot \psi_L^i) \chi_R; \quad \Psi_L^n = \frac{1}{2a} (\bar{\psi}_L^i \cdot \chi_R) \psi_L^i. \quad (10)$$

These fermion bound states possess the “wrong” chiralities in contrast with the “right” chiralities possessed by the elementary fields ψ_L^i and χ_R . The two-point Green functions of these charged Ψ_R^i and neutral Ψ_L^n have poles at the total momentum $p = \pi_A$ [17],

$$S_{MM}^{ij}(p) = \int_x e^{-ipx} \langle \Psi_R^i(0) \bar{\Psi}_R^j(x) \rangle \simeq \delta_{ij} \frac{\frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu}{\frac{1}{a^2} \sum_\mu \sin^2 p_\mu + M^2(p)} P_L; \quad (11)$$

$$S_{MM}^n(p) = \int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\Psi}_L^n(x) \rangle \simeq \frac{\frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu}{\frac{1}{a^2} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R, \quad (12)$$

$$M(p) = 8ag_2 w^2(p), \quad (13)$$

where $p \sim \pi_A$ and $w^2(p) \neq 0$. And the two-point Green functions for doublers ($p \sim \pi_A$) of the elementary fields χ_R and ψ_L^i are given by,

¹The soft symmetry breaking for the low-energy modes ($p \sim 0$) is allowed.

$$S_{LL}^{ij}(p) = \int_x e^{-ipx} \langle \psi_L^i(0) \bar{\psi}_L^j(x) \rangle \simeq \delta_{ij} \frac{\frac{i}{a} \gamma_\mu \sin(p)^\mu}{\frac{1}{a^2} \sin^2 p + M^2(p)} P_R; \quad (14)$$

$$S_{RR}(p) = \int_x e^{-ipx} \langle \chi_R(0) \bar{\chi}_R(x) \rangle \simeq \frac{\frac{i}{a} \gamma_\mu \sin(p)^\mu}{\frac{1}{a^2} \sin^2 p + M^2(p)} P_L. \quad (15)$$

The three-fermion-states (10,11,12) are Weyl fermions and respectively mix with the doublers of the elementary Weyl fields $\bar{\chi}_R$ and $\bar{\psi}_L^i$ (14,15),

$$S_{ML}^{ij}(p) = \int_x e^{-ipx} \langle \Psi_R^i(0) \bar{\psi}_L^j(x) \rangle \simeq \delta_{ij} \frac{M(p)}{\frac{1}{a^2} \sin^2 p + M^2(p)} P_R, \quad (16)$$

$$S_{MR}^n(p) = \int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\chi}_R(x) \rangle \simeq \frac{M(p)}{\frac{1}{a^2} \sin^2 p + M^2(p)} P_L. \quad (17)$$

As a result, the neutral Ψ_n and charged Ψ_c^i Dirac fermions are formed²,

$$\Psi_c^i = (\psi_L^i, \Psi_R^i); \quad \Psi_n = (\Psi_L^n, \chi_R), \quad (18)$$

and the spectrum is vector-like and massive. These show that all doublers are decoupled as very massive Dirac fermions consistently with the $SU_L(2) \otimes U_R(1)$ chiral symmetries, since the three-fermion-states (10) carry the appropriate quantum numbers of the chiral groups that accommodate ψ_L^i and χ_R .

III. THE MOMENTUM-DEPENDENT STATIC LIMIT AND ITS ANALYTICAL PROPERTY

Eqs.(11,12) for doublers is obtained by the the strong-coupling expansion in powers of $\frac{1}{g_2}$. For the strong coupling $g_2 \gg 1$, the kinetic terms can be dropped and the strong-coupling limit is given as,

$$\begin{aligned} Z &= \Pi_{x i \alpha} \int [d\bar{\chi}_R^\alpha(x) d\chi_R^\alpha(x)] [d\bar{\psi}_L^{i\alpha}(x) d\psi_L^{i\alpha}(x)] \exp(-S_2(x)) \\ &= (2g_2)^{4N} (\det \Delta^2(x))^{4N}, \quad g_2 \gg 1, \end{aligned} \quad (19)$$

where the determinant is taken only over the lattice space-time and N is the number of lattice sites. For the non-zero eigenvalues of the operator $\Delta^2(x)$ (2), which are associating to the doublers ($p \simeq \pi_A$) of $\psi_L^i(x)$ and $\chi_R(x)$, eq. (19) shows the existence of a sensible strong-coupling limit. Note that the operator $\Delta(x) \sim 2w(p)$ (2,19) has different eigenvalues $4 \sim 6$ with respect to different doublers ($p \sim \pi_A$), and the strong coupling expansion is actually in terms of powers of $\frac{1}{4g_2 w^2(p)}$. As the consequence, the two-point Green functions (11-17) computed by the strong coupling expansion with respect to doublers should be a good approximation even for the intermediate coupling

$$a^2 g_2 \sim O(1). \quad (20)$$

This discussion agrees with the qualitatively determined critical value $g_2^{c,a} = 0.124$ in (4), above which all doublers are decoupled via eqs.(18).

However, as for the zero eigenvalues of the operator $\Delta^2(x)$, which precisely correspond to the low-energy modes ($p \sim 0$) of $\psi_L^i(x)$ and $\chi_R(x)$, this strong-coupling limit is trivial and the strong-coupling expansion in powers of $\frac{1}{g_2}$ breaks down. This physically means the weakness of the effective multifermion coupling for such low-energy modes of $\psi_L^i(x)$ and $\chi_R(x)$. Nevertheless, we cannot exclude the possibility of low-energy modes of three-fermion-states (10), which are represented by the poles $p \sim 0$ of the propagators (11,12).

On the other hand, as a consequence of the multifermion interacting action (1) being local, in the strong coupling limit the effective action (inverse propagator), which is bilinear in terms of interpolating fields, should be local and analytical in the whole Brillouin zone. Thus, the “no-go” theorem of Nielsen and Ninomiya is still applicable to this case [21]. Based on such an observation, one might argue the existence of the massless spectrum of the charged and neutral three-fermion-states (10) by the analytic continuation of their propagators (11,12) from $p \sim \pi_A$ to $p \sim 0$. As a result, the low-energy spectrum (18) is also vector-like.

²The propagators of these Dirac fermions can be obtained by summing all relevant Green functions shown in above.

Indeed, due to the locality of action (1) presented in this paper, all Green functions must be analytically continuous functions in energy-momentum space, provided the dynamics is fixed by given g_1 and g_2 . In the strong coupling symmetric phase (PSM) where $g_1 \gg 1$ and a sensible strong-coupling limit exists, two-point Green functions for three-fermion-states (10) are essentially indistinguishable from that of the elementary fermion fields $\psi_L^i(x)$ and $\chi_R(x)$ appearing in the Lagrangian. In such a phase, the analytical continuation of Green functions for both elementary and composite fields in the whole momentum space does result in the “vector-like” phenomenon, as asserted by the “no-go” theorem for a free-fermion theory. The only loophole would appear if the propagators of interpolating fields (three-fermion-states) properly vanished and no longer had poles at $p \sim 0$. This indicates that at $p \sim 0$, these three-fermion-states dissolve into three-fermion-cuts, where the “no-go” theorem is entirely inapplicable. We discuss the concept of three-fermion-cuts in the next section.

IV. GENERALIZED FORM-FACTORS OF THREE-FERMION-STATES

We turn to discuss how these three-fermion-states (12,11) with the “wrong” chiralities dissolve into three-fermion-cuts in segment \mathcal{A} for the low-energy limit ($p \rightarrow 0$). These three-fermion-cuts [18,23]:

$$\mathcal{C}[\Psi_L^n(x)], \quad \mathcal{C}[\Psi_R^i(x)], \quad (21)$$

are the virtual states of three individual chiral fermions with a continuous energy spectrum, provided the total momentum p is fixed. However, these virtual states carry exactly the same quantum numbers and total momentum p as that of three-fermion-states. Thus, gauge symmetries are preserved in such a phenomenon of dissolving. The dynamics of the three-fermion-states dissolving into their virtual state is that the negative binding energy of three-fermion-states goes to zero. In the energy plane, it was shown that due to the variety of effective interactions (potential), the poles for bound states can be analytically continued to the cuts for virtual states on the physical sheet [24]. Presumably, in this analytical property of effective interactions, no other dynamics, e.g., spontaneous symmetry breaking, takes place. In segment \mathcal{A} , the weakness of effective multifermion couplings for the low-energy modes of ψ_L^i and χ_R could lead to the vanishing of the binding energy of the three-fermion-states (10). We can conceive a “dissolving” scale (threshold) ϵ

$$\tilde{v} \ll \epsilon < \frac{1}{a}, \quad (22)$$

where \tilde{v} is the possible soft spontaneous symmetry breaking scale ($a\tilde{v} \simeq 0$). From inequality (22), we understand that no hard spontaneous symmetry breaking in segment \mathcal{A} is extremely crucial for the possibility of analytical continuation of propagators from poles for three-fermion-states to cuts for virtual states of three individual fermions.³

In the relativistic Lagrangian approach, to discuss the property of three-fermion-states dissolving into three-fermion-cuts, we are bound to dynamically calculate two-point functions of three-fermion-states (Fig.1) to identify not only their poles, but also the corresponding residues. Using the strong coupling expansion, we approximately determined the simple poles for doublers ($p \sim \pi_A$) in eqs.(11,12). The residues $Z_R(p)$ and $Z_L(p)$ of these simple poles are defined as,

$$S_{MM}^{ij}(p) = \int_x e^{-ipx} \langle \Psi_R^i(0) \bar{\Psi}_R^j(x) \rangle \simeq \delta_{ij} \frac{Z_R(p) \frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu Z_R(p)}{\frac{1}{a^2} \sum_\mu \sin^2 p_\mu + M^2(p)} P_L; \quad (23)$$

$$S_{MM}^n(p) = \int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\Psi}_L^n(x) \rangle \simeq \frac{Z_L(p) \frac{i}{a} \sum_\mu \sin p^\mu \gamma_\mu Z_L(p)}{\frac{1}{a^2} \sum_\mu \sin^2 p_\mu + M^2(p)} P_R. \quad (24)$$

In fact, these residues represent the generalized form factors of three-fermion-states. The $Z_{L,R}(p)$ momentum dependence indicates that different doublers have different form factors, which implies the “size” of bound states is different from one doubler to another. This is clearly attributed to the momentum dependence of effective multifermion couplings in action (1). If these residues $Z_{L,R}(p = \pi_A)$ are positive⁴ and finite constants with respect to each doubler, we can just make a wave-function renormalization of three-fermion-states with respect to each doubler,

³At the dissolving scale $\epsilon \gg \tilde{v}$, we can approximately treat elementary massive fermions (e.g. massive particles in the Standard Model) as massless particles.

⁴We do not want to have ghost states with negative norm.

$$\Psi_R^i|_{ren} = Z_R^{-1}\Psi_R^i; \quad \Psi_L^n|_{ren} = Z_L^{-1}\Psi_L^n, \quad (25)$$

and the two-point Green functions (23,24) turn into eqs.(11,12) in terms of the renormalized fields (25).

The residues (generalized form factors) $Z_{R,L}(p)$ (23,24) of the three-fermion-states (10) are given by one-particle irreducible (1PI) truncated Green functions (see Fig.2),

$$Z_L(p) = \int_x e^{-ipx} \frac{\delta^{(2)}\Gamma}{\delta\Psi_L'^n(x)\delta\bar{\chi}_R'(0)}, \quad Z_R(p) = \int_x e^{-ipx} \frac{\delta^{(2)}\Gamma}{\delta\Psi_R^i(x)\delta\bar{\psi}_L^{tj}(0)}. \quad (26)$$

The “primed fields” $\Psi_R'^n(x)$ and $\Psi_L^i(x)$ of three-fermion-states are defined by eqs.(41) and (42) in ref. [17] through the generating functional approach. These generalized form factors $Z_L(p)$ and $Z_R(p)$ give the overlap between three-fermion-operators ($\Psi_R^i(x)$, $\Psi_L^n(x)$) and the interpolating three-fermion-states, that appear in the space of asymptotic states of the theory in the scaling region.

This description coincides with the renormalization of n-point 1PI functions with insertions of composite operators. In general, the renormalized n-point 1PI functions $\Gamma_{ren}^{(n)}$ with single and two insertions of composite operators are given by [25],

$$\begin{aligned} \Gamma_{ren}^{(n)}(p_1, q_1, q_2, \dots, q_n) &= Z\Gamma_{reg}^{(n)}(p_1, q_1, q_2, \dots, q_n), \\ \Gamma_{ren}^{(n)}(p_1, p_2, q_1, q_2, \dots, q_n) &= Z^2\Gamma_{reg}^{(n)}(p_1, p_2, q_1, q_2, \dots, q_n), \end{aligned} \quad (27)$$

where $\Gamma_{reg}^{(n)}$ are the regularized n-point 1PI functions and p_1 and p_2 stand for the momenta entering the composite operators. Similarly given by eq.(26) (Fig.2) [25], the Z ’s are the generalized “wave-function renormalizations” of composite operators. It is worthwhile to emphasize that for residues $Z_{L,R}$ being positively finite, the wave-function renormalization of composite fields is the exactly same as the wave-function renormalization of elementary fields. In fact, composite particles are indistinguishable from elementary particles in this case. However, the normal wave-function renormalizations of elementary fields appearing in the Lagrangian is attributed to the fact that these elementary fields are defined at different scales rather than their “form factor”. Note that the elementary fields $\psi_L^i(x)$ and $\chi_R(x)$ in the action (1) are bare fields and have not yet been renormalized.

However, in the analytical continuation of the propagators (23,24) in momentum space, let us assume an interesting case that the residues $Z_{L,R}(p)$ positively vanish in the limit $p \rightarrow 0$ for the pole $p \sim 0$ in eqs.(23, 24),

$$Z_{L,R}(p) \rightarrow O(p^n) \quad p \rightarrow 0, \quad (28)$$

with $n = 2, 4, 6, \dots$. This property of $Z_{L,R}(p)$ could be realized by an appropriate momentum-dependence of the effective multifermion couplings g_1, g_2 . Eq.(28) implies that $p \sim 0$ is no longer a pole for a relativistic particle in the propagators (23, 24). We are not allowed to make wave-function renormalization (25) with respect to $p \sim 0$. Eq.(28) indicates that the “size” of bound states (10) increases as $p \rightarrow 0$. These three-fermion-states may eventually dissolve into the virtual states of three individual fermions, whose possible configuration in momentum space is $(p, p, -p)$ [18,26], where p is the total momentum and the relative momentum (q) is zero. This dissolving phenomenon is entirely determined by both the dynamical and kinetic properties of the interacting theory.

This is reminiscent of the papers [27] discussing whether helium is an elementary or composite particle based on the vanishing of wave-function renormalizations of composite states. It is normally referred to as the composite condition that the wave-function renormalizations of bound states go to zero ($Z \rightarrow 0$) [28]. So far, we only give an intuitive and qualitative discussion of the dissolving phenomenon on the basis of the relations between the residues (generalized form factors) $Z_{L,R}(p)$, renormalized three-fermion-states and virtual states of three individual fermions (three-fermion-cut). Evidently, we are bound to do some dynamical calculations to show this phenomenon could happen.

V. THE THREE-FERMION-CUT FOR NEUTRAL CHANNEL

The form factor $Z_L(p)$ in eq.(26) for the left-handed three-fermion-state Ψ_L^n can be completely determined by the Ward identity (6). We appropriately take functional derivative of the Ward identity (6) with respect to the “primed” field $\Psi_L'^n(x)$, and we obtain,

$$Z_L(p) = aM(p), \quad (29)$$

which are positive and finite constants for $p = \pi_A$ (see eq.(13)). Together with the propagators (12) obtained by the strong coupling expansion for $g_2 \gg 1$ and $p \sim \pi_A$, we conclude that the doublers of the neutral channel (24) are indeed relativistic massive particles, whose wave functions can be renormalized according to (25).

Given the strong coupling $g_2 \gg 1$, in spite of the propagator (24) for the neutral three-fermion-state Ψ_L^n resulting from the strong coupling expansion for $p \sim \pi_A$, we can make an analytical continuation from $p \sim \pi_A$ to $p \sim 0$. However, for the reason that $Z_L^2(p) \rightarrow O(p^8)$ as $p \rightarrow 0$, we cannot conclude that $p \sim 0$ is a simple pole for a relativistic massless particle by an analytical continuation of the propagator (24) from $p \sim \pi_A$ to $p \sim 0$, whereas we are not allowed to perform a wave-function renormalization (25) for $p \sim 0$. Actually, the propagator (24) is vanishing at $p \sim 0$. It is important to point out that at the limit of $p \rightarrow 0$, the vanishing of the propagator (24) is definitely positive, i.e., it is a double zero. This implies that ghost states with negative norm would not appear in low-energy spectrum.

However, on the other hand, we cannot conclude that the vanishing of the neutral propagator (24) for $p \sim 0$ indicates the virtual state of three fermions in the neutral channel. Since at $p = 0$ the static limit is trivial, the computation of the propagator (24) by the strong coupling expansion may not be reliable. We must compute exactly the same two-point Green function for the neutral three-fermion-operator,

$$\int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\Psi}_L^n(x) \rangle, \quad (30)$$

for $g_2 \gg 1$ and $p \sim 0$, as that (24) computed by the strong coupling expansion for $p \sim \pi_A$ and $g_2 \gg 1$. Since the coupling g_2 in segment \mathcal{A} can be arranged such that

$$a^2 g_2 w(p) < 1, \quad \infty > g_2 \gg 1, \quad p \sim 0, \quad (31)$$

we can adopt the effective weak coupling expansion in powers of $a^2 g_2 w(p)$ to calculate the Green function (30). This is due to the fact that χ_R is always an external field in Feynman diagrams (see Fig.3) of computing this Green function (30) and each χ_R is associated with $g_2 w(p)$. This is a very reliable approximation as far as condition (31) holds. Here, we want to emphasize that the meaning of $p \sim 0$ is,

$$\epsilon > p \gg a\tilde{v} \simeq 0. \quad (32)$$

We can determine the value of g_2 in such a way that the threshold scale ϵ is much larger than the soft symmetry breaking scale \tilde{v} .

Clearly, no bound states in eq.(30) can be formed for such a weak effective coupling. Only the virtual state for the neutral three-fermion-cut (21) can be found in this Green function (30). We show this by calculating (30) in the effective weak coupling expansion as indicated in the Feynman diagrams (Fig.3). The leading order ($O(1)$) of this expansion is the first diagram in Fig.3, which corresponds to two parts,

$$W_\circ(x) = - \left(\frac{1}{2a} \right)^2 \langle \psi_L^{i\gamma}(0) \bar{\psi}_L^{j\delta}(x) \rangle \langle \psi_R^\delta(0) \bar{\psi}_R^\gamma(x) \rangle \langle \psi_L^{j\alpha}(0) \bar{\psi}_L^{i\beta}(x) \rangle, \quad (33)$$

$$W'_\circ(x) = \left(\frac{1}{2a} \right)^2 \langle \psi_L^{i\gamma}(0) \bar{\psi}_R^\gamma(x) \rangle \langle \psi_R^\delta(0) \bar{\psi}_L^{j\delta}(x) \rangle \langle \psi_L^{j\alpha}(0) \bar{\psi}_L^{i\beta}(x) \rangle, \quad (34)$$

where $\gamma, \delta, \beta, \alpha$ are spinor indices. In momentum space, eqs.(33) and (34) are given as,

$$W_\circ(p) = - \int_{qk} S_{LL}^{ji}(p+q) \text{tr} \left[S_{RR}(k - \frac{q}{2}) S_{LL}^{ij}(k + \frac{q}{2}) \right] \left(\frac{1}{2a} \right)^2, \quad (35)$$

$$W'_\circ(p) = \int_{qk} S_{LL}^{ji}(p+q) \text{tr} \left[\Sigma^i(k - \frac{q}{2}) \right] \text{tr} \left[\Sigma^j(k + \frac{q}{2}) \right] \left(\frac{1}{2a} \right)^2, \quad (36)$$

where p is the total external momentum, k and q are the relative internal momenta. Since segment \mathcal{A} is an entirely symmetric phase, i.e. $\Sigma^i(k) = 0$ (see eqs.(8,9)), eq.(36) identically vanishes. In general, we can write the internal propagators $S_{LL}^{ij}(k)$ and $S_{RR}(k)$ in eq.(35) as follow,

$$S_{LL}^{ij}(k) = \delta_{ij} f_L(k^2) \gamma_\mu \sin k^\mu P_R, \quad (37)$$

$$S_{RR}(k) = f_R(k^2) \gamma_\mu \sin k^\mu P_L. \quad (38)$$

These internal propagators $S_{LL}^{ij}(k)$ and $S_{RR}(k)$ with internal momentum $k \in (0, \pi_A]$ can be given by eqs.(14,15) calculated by the strong coupling expansion in segment \mathcal{A} . We define,

$$R(k, q) = \text{tr} \left[S_{RR}(k - \frac{q}{2}) S_{LL}^{ij}(k + \frac{q}{2}) \right], \quad (39)$$

where we ignore indices i, j in the LHS. Since $R(k, q)$ (39) is an even function with respect to k and q ,

$$R(k, q) = R(-k, q), \quad R(k, q) = R(k, -q), \quad (40)$$

one can easily show for the external momentum $p \sim 0$,

$$W_o(p) \simeq ac_o(i\gamma_\mu p^\mu) + O(p^2), \quad (41)$$

where c_o is a constant.

The second diagram of Fig.3 denotes the contribution of order $O[(g_2 w(p))^2]$ to the Green function (30) given by⁵,

$$W_1(p) = [g_2 w(p)]^2 V(p) \tilde{S}_{RR}(p) V(p), \quad (42)$$

where $\tilde{S}_{RR}(p)$ is the full propagator of χ_R , as indicated by a full circle in the middle of the Feynman diagram (Fig.3), which is summed over all contributions of this effective weak coupling expansion (Fig.4). We can adopt eq.(15) for $\tilde{S}_{RR}(p)$ by an analytical continuation from $p \sim \pi_A$ to $p \sim 0$. On the other hand, as a consequence of the χ_R -shift-symmetry (7), $\tilde{S}_{RR}(p)$ for $p \sim 0$ is a free propagator,

$$\tilde{S}_{RR}(p) \simeq \frac{a}{i\gamma_\mu p^\mu}. \quad (43)$$

In eq.(42), $V(p)$ is given by

$$V(p) = \int_{qk} 4w(k - \frac{q}{2}) S_{LL}^{ji}(p + q) \text{tr} \left[S_{RR}(k - \frac{q}{2}) S_{LL}^{ij}(k + \frac{q}{2}) \right], \quad (44)$$

where the factor $4w(k - \frac{q}{2})$ comes from the interaction vertex. Using a similar analysis to that giving eq.(41), we get for $p \sim 0$ (up to a finite constant),

$$V(p) \sim -a(i\gamma_\mu p^\mu), \quad (45)$$

which cancels the pole of eq.(43) at $p \sim 0$. As a result, we obtain ($w(p) \sim p^2$, $p \sim 0$),

$$W_1(p) \simeq (a^2 g_2 p^2)^2 ac_1(i\gamma_\mu p^\mu), \quad (46)$$

where c_1 is a finite number. Thus, for $g_2 \gg 1$ and $p \sim 0$, the Green function (30) is,

$$\int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\Psi}_L^n(x) \rangle \simeq W_o(p) + W_1(p), \quad (47)$$

which is regular at $p \sim 0$. This shows that the neutral channel is a virtual state for the three-fermion-cut in region (31). Obviously, it is absolutely incorrect for doublers ($p \sim \pi_A$) since the effective coupling (31) can be extremely large, bound states $\Psi_L^n(x)$ (10) must be formed.

Note that in this region (31), the mixing between the elementary field χ_R and neutral three-fermion-state Ψ_L^n (10) calculated by the strong coupling expansion for $p \sim \pi_A$ is given by (17). Analytical continuation of (17) from $p \sim \pi_A$ to $p \sim 0$ shows the vanishing of the mixing at $p \sim 0$, while this mixing gives rise to the gauge-invariant mass terms for doublers $p \sim \pi_A$. On the other hand, the mixing (17) can be calculated by the effective weak coupling expansion (31). The leading order is,

$$\int_x e^{-ipx} \langle \Psi_L^n(0) \bar{\psi}_R(x) \rangle = \int_x e^{-ipx} \left(\frac{1}{a} \right) \langle \psi_L^i(0) \bar{\psi}_R(x) \rangle \langle \psi_R(0) \bar{\psi}_L^i(0) \rangle + O(g_2 w(p)), \quad (48)$$

which vanishes for no hard spontaneous symmetry breaking.

Based on the computations of the Green function for the neutral channel (30) at both $p \sim \pi_A$ (24) and $p \sim 0$ (47) for $g_2 \gg 1$, we conclude that in the intermediate region (31) of the gauge-symmetric segment \mathcal{A} , the low-energy spectrum ($p \sim 0$) is

⁵The contribution (34) is zero, and other possible contributions are identically zero, owing to $\Sigma(k) = 0$.

- undoubled for all doublers ($p \sim \pi_A$) decoupled;
- chiral because for only exists a free right-handed mode χ_R , while Ψ_L^n is no longer a bound state, it is instead a virtual state $\mathcal{C}[\Psi_L^n]$.

By the analytical continuity of the Green function (propagator) (30) in terms of the total momentum p , from eq.(24) for $p \sim \pi_A$ to eq.(47), we must find the threshold scale ϵ (22), where the neutral three-fermion-state $\Psi_L^n(x)$ (10) with the “wrong” chirality dissolves into its virtual states and only χ_R remains as a relativistic particle at $p \sim 0$. We emphasize that this already violates the “no-go” theorem even though the chiral fermion is neutral. However, it is worthwhile to point out that such a mechanism violating the “no-go” theorem is only expected to work in the cases of neutral and anomaly-free theories.

VI. THE THREE-FERMION-CUT FOR THE CHARGED CHANNEL

The form factor $Z_R(p)$ (26) for the right-handed three-fermion-state $\Psi_R^i(x)$ (10) can not be determined by the Ward identity (6). Instead, it can be calculated by using the results obtained in the strong coupling expansion for $g_2 \gg 1$ and $p \sim \pi_A$.

The 1PI-vertex function associated to $Z_R(p)$ is given by the truncated Green function (26) that is defined as (as indicated in Fig.2),

$$\begin{aligned} Z_R(p) &= \int_x e^{-ipx} \frac{\delta^{(2)}\Gamma}{\delta\Psi_R^i(0)\delta\bar{\psi}_L^j(x)} \\ &= \int_x e^{-ipx} \int_{z_1, z_2} \left(G_{MM}^{jl}(x, z_2)\right)^{-1} G_{ML}^{lk}(z_2, z_1) \left(G_{\text{free}}^{ki}(z_1, 0)\right)^{-1} \\ &= \left(S_{MM}^{jl}(p)\right)^{-1} S_{ML}^{lk}(p) \left(S_{\text{free}}^{ki}(p)\right)^{-1}, \end{aligned} \quad (49)$$

where

$$\begin{aligned} G_{MM}^{jl}(x, z_2) &= \langle \Psi_R^j(x) \bar{\Psi}_R^l(z_2) \rangle \rightarrow S_{MM}^{jl}(p) \\ G_{ML}^{lk}(z_2, z_1) &= \langle \Psi_R^l(z_2) \bar{\psi}_L^k(z_1) \rangle \rightarrow S_{ML}^{lk}(p') \\ G_{\text{free}}^{ki}(z_1, 0) &= \langle \psi_L^k(x) \bar{\psi}_L^i(z_1) \rangle \rightarrow S_{\text{free}}^{ki}(p''), \quad g_1, g_2 = 0, \end{aligned} \quad (50)$$

and their transformations into the momentum space in the last line of eq.(49).

Adopting the results $S_{MM}^{jl}(p)$ (11), $S_{ML}^{ij}(p)$ (16) and the free propagator $S_{\text{free}}^{ki}(p)$ of the ψ_L^i , we get,

$$Z_R(p) = aM(p), \quad (51)$$

which is the same as $Z_L(p)$ (29) directly derived from the Ward identity (6). Eq.(51) is a positive and finite constant for doublers $p \simeq \pi_A$ (see eq.(13)). Together with the propagator (23) obtained by the strong coupling expansion for $g_2 \gg 1$ and $p \sim \pi_A$, we conclude that the doublers of charged channel (23) are indeed relativistic massive particles, whose wave functions can be renormalized according to (25).

In spite of eqs.(23) and (51) obtained by the strong coupling ($g_2 \gg 1$) expansion for $p \sim \pi_A$, we can analytically continue the momentum “ p ” in these equations to the limit of $p \rightarrow 0$. When $p \rightarrow 0$, $Z_R^2(p) \rightarrow O(p^8)$, the propagator (23) of charged three-fermion-state vanishes. This implies that the low-energy state ($p \sim 0$) of $\Psi_R^i(x)$ with the “wrong” chirality is no longer a simple pole as a relativistic particle, instead, it is a virtual state,⁶ and the low-energy state of chiral fermions with “right” chirality should be realized in the spectrum of charged particles. We have to confess that unlike neutral channel case, without further more rigorous computations, numerical one for instance, we can not demonstrate such a scenario for realizing charged lattice chiral fermions to be definite ture, since the static limit is broken at $p = 0$. However, it is strongly emphasized that for the same reason that the static limit is broken at $p = 0$,

⁶We emphasize again that at the limit of $p \rightarrow 0$, the vanishing of the propagator of the charged three-fermion-state is definitely positive, i.e., it is a double zero, which implies that ghost states with negative norm do not appear and couple to the gauge field in the low-energy limit. Otherwise, the theory would be inconsistent [31].

we can not conclude the spectrum to be definitely vector-like as it was concluded in ref. [21]. What we can conclude is a loophole at $p \sim 0$ opening up. We go further in following to check in more details.

In order to directly show that the low-energy state ($p \sim 0$) of $\Psi_R^i(x)$ is the virtual state for a three-fermion-cut. We have to compute exactly the same two-point Green function,

$$\int_x e^{-ipx} \langle \Psi_R^i(0) \bar{\Psi}_R^j(x) \rangle, \quad (52)$$

of the charged channel for $p \sim 0$ and $g_2 \gg 1$ as that (23) computed by the strong coupling expansion for $p \sim \pi_A$ and $g_2 \gg 1$. However, unlike the case of neutral channel, we do not have the reliable method of the effective weak coupling expansion in powers of $a^2 g_2 w(p)$ (31), since $\psi_L^i(x)$ is always an external field in the Feynman diagrams computing the Green function (52) and each ψ_L does not associate with $g_2 w(p)$. In this case, the effective weak coupling expansion used for computing the neutral channel (Fig.3) must breakdown for $g_2 \gg 1$.

Nevertheless, we observe that the last binding threshold is located at $a^2 g_c^a = 0.124$ (4). Above this threshold ($g_2 > g_c^a$, $a^2 g_2 \sim O(1)$ eq.(20)), all doublers are supposed to be decoupled via eqs.(11,12) and (13,18). The critical point $a^2 g_c^a = 0.124$ is a rather small number. Thus, at $a^2 g_2 \sim O(1)$, we introduce N_f , an addition number of fermion flavours of ψ_L^i and χ_R so that,

$$a^2 g_2 > 0.124, \quad \tilde{g}_2 = a^2 g_2 N_f < 1 \quad \text{fixed}, \quad N_f = 3 \sim 8, \quad (53)$$

where the value of N_f depends on the value of $a^2 g_2$ considered. Therefore, in a certain intermediate region of $a^2 g_2 \sim O(1)$, we adopt the large- N_f technique to control the convergence of the approximation (see Fig.3) and calculate the Green function (52) of the charged three-fermion-operators for the low-energy $p \sim 0$. Hence we can get a qualitative insight into the charged low-energy spectrum ($p \sim 0$) within the intermediate region of g_2 (53). We expect that the dynamics of the interaction is not greatly changed by introducing more flavours. As for $a^2 g_2 > 1$, such large- N_f technique is doomed to fail.

Analogous to the case of neutral channel (41), the non-trivial leading order $O(N_f)$ contribution, as indicated by the first diagram in Fig.3, is ($p \sim 0$),

$$W_o^c(p) = -N_f \int_{qk} S_{RR}(p+q) R(k, q) \left(\frac{1}{2a} \right)^2. \quad (54)$$

The second order $(a^2 g_2 N_f)^2$ contribution, as indicated by the second diagram in Fig.3, is ($p \sim 0$),

$$W_1^c(p) = (g_2 N_f)^2 V^c(p) \tilde{S}_{LL}(p) V^c(p) \quad (55)$$

where we ignore indices ij and $V^c(p)$ is given as ($p \sim 0$)

$$V^c(p) = - \int_{qk} 4w(p+q)w(k - \frac{q}{2}) S_{RR}(p+q) R(k, q), \quad (56)$$

where the factor $4w(p+q)w(k - \frac{q}{2})$ comes from interacting vertices. The full propagator $\tilde{S}_{LL}(p)$ of $\psi_L^i(x)$ in eq.(55), as indicated by a full circle in middle of the second diagram in Fig.3, is calculated by the train approximation (as indicated in Fig.4) for the external total momentum $p \sim 0$,

$$\tilde{S}_{LL}^{ij}(p) = \int_x e^{-ipx} \langle \psi_L^i(0) \bar{\psi}_L^j(x) \rangle \simeq Z_2^{-1}(p) S_{LL}^{ij}(p). \quad (57)$$

The wave-function renormalization $Z_2(p)$ of the elementary field $\psi_L^i(x)$ is given by (see Appendix A),

$$Z_2(p) = 1 + \frac{(\tilde{g}_2)^2}{N_f} \int_{k,q} \left(4w(p+q)w(k - \frac{q}{2}) \right)^2 S_{RR}(p+q) S_{LL}(p) R(k, q). \quad (58)$$

Note that in eqs.(54,56,58) and $R(k, q)$ (39), the internal propagators $S_{LL}(k)$ for $\psi_L^i(x)$, and $S_{RR}(k)$ for $\chi_R(x)$ are respectively given by eqs.(37,38) or approximately by eqs.(14,15). The reasons for such choices are that in the region (53), doublers ($k \sim \pi_A$) are supposed to be decoupled via eq.(14,15) and the internal momentum k runs from $k \sim 0$ to $k \sim \pi_A$. As for the propagator $S_{LL}(p)$ in eqs.(57,58) for the external total momentum $p \sim 0$, we adopt eq.(14) by analytical continuation from $p \sim \pi_A$ to $p \sim 0$.

Analogous to that (41) in the neutral channel, for the total momentum $p \sim 0$, the leading order contribution (54) becomes⁷,

$$W_o^c(p) \sim aN_f(i\gamma_\mu p^\mu). \quad (59)$$

As for the second order contribution (55), we find for $p \sim 0$,

$$V^c(p) \sim a(i\gamma_\mu p^\mu), \quad (60)$$

which cancels the pole at $p = 0$ stemming from the propagator \tilde{S}_{LL} in eq.(57). And the wave-function renormalization $Z(p)$ of the elementary fields $\psi_L^i(x)$ at $p = 0$ is given by,

$$Z_2(0) = 1 + \text{const.}, \quad (61)$$

due to eq.(39). This indicates that the relativistic particle ($p = 0$) of ψ_L^i receives a wave-function renormalization $Z_2(0)$. As a result, the second-order contribution (55) reads,

$$W_1^c(p) \sim (\tilde{g}_2)^2 a(i\gamma_\mu p^\mu). \quad (62)$$

Thus, for $p \sim 0$ the Green function (52) is approximately computed as,

$$\int_x e^{-ipx} \langle \Psi_R^c(0) \bar{\Psi}_R^c(x) \rangle \simeq W_o^c(p) + W_1^c(p), \quad (63)$$

which is regular at $p \sim 0$. This implies that in this intermediate region (53), the charged channel of three-fermion-operators at $p \sim 0$ is not a simple pole for a massless relativistic particle with the “wrong” chirality, rather it is regular for a virtual state of three-fermion-cut. This agrees with the result obtained by analytical continuation of the propagator (23) from $p \sim \pi_A$ to $p \sim 0$.

An important consistent check is to examine the equation (55) for the “doublers”, i.e., the external momentum $p \sim \pi_A$. In fact, for $p \sim \pi_A$,

$$V^c(p) \simeq -4w(\pi_A) \int_{qk} w(k - \frac{q}{2}) S_{RR}(p+q) R(k, q). \quad (64)$$

This results in the coupling \tilde{g}_2 in the second-order contribution (55) being enhanced up a factor of $w^2(\pi_A)$ (see eq.(2)),

$$W_1^c(p) \sim (\tilde{g}_2 w(\pi_A))^2 V^c(p) \tilde{S}_{LL}(p) V^c(p). \quad (65)$$

Analogously, for $p \sim \pi_A$,

$$Z_2(p) \simeq 1 + \frac{2(\tilde{g}_2 w(\pi_A))^2}{N_f} \int_{k,q} \left(4w(k - \frac{q}{2})\right)^2 S_{RR}(p+q) S_{LL}(p) R(k, q). \quad (66)$$

The consequence is the complete breakdown of the large- N_f expansion (53), which is not convergent for $p \sim \pi_A$, to calculate the two-point Green function (52) of charged three-fermion-operators. This implies that bound states (three-fermion-states) with the total momentum $p \sim \pi_A$ should be formed, consistently with the bound states that we find by the strong coupling expansion for $p \sim \pi_A$.

We turn to the computation of Green function for the mixing between the elementary field ψ_L^i and charged three-fermion-state Ψ_R^j (10). This mixing can be calculated by the large- N_f expansion (53). The non-trivial leading order $O(N_f)$ is explicitly written as,

$$\int_x e^{-ipx} \langle \Psi_R^i(0) \bar{\psi}_L^j(x) \rangle = \int_x e^{-ipx} \left(\frac{1}{a} \right) \langle \psi_R(0) \bar{\psi}_L^j(x) \rangle \langle \psi_L^i(0) \bar{\psi}_R(0) \rangle + O\left(\frac{\tilde{g}_2}{N_f}\right), \quad (67)$$

where $O\left(\frac{\tilde{g}_2}{N_f}\right)$ stands for higher order contributions. Eq.(67) vanishes for non spontaneous symmetry breaking (see eq.(8)). Consistently, an analytical continuation of the Green function (16) from $p \sim \pi_A$ to $p \sim 0$ shows vanishing of the mixing at $p \sim 0$ as well.

⁷“ \sim ” indicates up to a finite constant.

Finally we check that in the intermediate region of g_2 (53), whether the Green function (57) for the elementary field ψ_L^i has a simple pole at $p \sim 0$ representing a relativistic massless particle. Due to $Z_2(0) = 1 + \text{const.}$ (61), one can check the propagator (57) for ψ_L^i has a simple pole at $p = 0$, which indicates a massless, charged left-handed ψ_L^i in the low-energy spectrum. This agrees with the analytical continuation of the propagator (14) from $p \sim \pi_A$ to $p \sim 0$.

In the continuation from eq.(23) for $p \sim \pi_A$ to eq.(63) for $p \sim 0$, we must meet the “dissolving” threshold ϵ , which should be the same as that in the neutral channel. We need to point out that the computation of charged channel is different from the computation of neutral channel. In the neutral channel, the Green functions (15,17,24) for $p \sim \pi_A$ and (43,47,48) for $p \sim 0$ are both consistently calculated in $g_2 \gg 1$. However, in the charged channel, the Green functions (57,63, 67) for $p \sim 0$ are computed in the intermediate region $a^2 g_2 \sim O(1)$ (53), while the same Green functions (14,23,16) for $p \sim \pi_A$ are computed in the region $g_2 \gg 1$. This may raise a question whether the dynamics we explore for $p \sim 0, a^2 g_2 \sim O(1)$ and for $p \sim \pi_A, a^2 g_2 \gg 1$ are consistent. We argue that such studies are qualitatively justified, since the computations in the strong coupling expansion with respect to doublers ($p \sim \pi_A$) are valid as well for $a^2 g_2 \sim O(1)$ (20) as discussed in the section II.

In conclusion, on the basis of approximate calculations of relevant two-point Green functions of elementary and composite three-fermion-operators with respect to doublers $p \sim \pi_A$ and low-energy mode $p \sim 0$, we qualitatively explore a possible scaling window in segment \mathcal{A} , where in the low-energy spectrum, the three-fermion-states with the “wrong” chirality turn into their corresponding virtual states, and elementary fermion states with the “right” chirality remain as massless states. Evidently, full non-perturbative numerical simulation to explore this scaling window is very inviting and necessary in particular for any solid conclusions in the charged channel.

VII. DISCUSSIONS OF RELATIONS BETWEEN OUR MODEL, THE EICHTEN-PRESKILL AND SMIT-SWIFT MODELS

Multifermion coupling models, for instance the Eichten-Preskill model and the one presented in this paper, can formally be rewritten as scalar-fermion coupling models in the lattice (cutoff) scale by introducing appropriate scalar fields, which mimic the bound states of two fermions. There are many classes of multifermion coupling models and scalar-fermion couplings models, the multifermion coupling models and scalar-fermion coupling models belong to the same universal class and are certainly equivalent in the sense that not only they can be formally translated each other by introducing auxiliary scalar fields in path integrals, but also they share the same infrared fix (scaling) points and lines in the phase space and have the same low-energy spectra, couplings and relevant operators upon those fix points and lines. The later property is more important and difficult to generally prove the equivalence between a given multifermion model and a given scalar-fermion coupling model. What we can do for a given either multifermion coupling model or scalar-fermion coupling model is to analyze all possible infrared scaling region in phase space.

The Smit-Swift model certainly belongs to one universal class of scalar-fermion coupling models and equivalent multifermion coupling models. The Eichten-Preskill multifermion-coupling model and its equivalent scalar-fermion coupling model could belong to another universal class that is not proved to be the same as the universal class of models of the Smit-Swift type. It is no doubt that our multifermion-coupling model has its equivalent scalar-fermion coupling model. However, no one has demonstrated that our multifermion-coupling model must be in the same universal classes of either the Smit-Swift model or the Eichten-Preskill model. The only thing to do is to analyze all possible scaling region of our model by appropriate methods as that we present in this paper. We can clearly point out why and what could make possible differences between our model and the Eichten-Preskill as well as Smit-Swift models in the existence of a possible scaling region for lattice chiral fermions.

The common feature of our model, the Eichten-Preskill model and Smit-Swift model is the momentum-dependence of multifermion couplings or scalar-fermion couplings attempting to give doublers $p = \pi_A$ heavy masses and decouple them from the low-energy spectrum. In the phases of weak multifermion couplings or scalar-fermion couplings, doublers are not decoupled and the spontaneous symmetry breaking could occur. Therefore, in these three models we only need to discuss the phase of strong multifermion couplings or scalar-fermion couplings where three-fermion-states or bound states of scalar-fermion are formed, and spectrum is expected to be vector-like. Question clearly is whether we can find a scaling region in the strong-coupling phase of these three models, and in this scaling region the low-energy mode of three-fermion-states to scalar-fermion bound states with the “wrong” chirality can be excluded, and the chiral fermions with the “right” chirality exist in the low-energy spectrum.

In the strong-coupling phase, the momentum-dependence of the strong multifermion coupling of our model is elaborately made in such a way that in a segment of the strong coupling phase ($\mathcal{A}g_1 = 0, g_2 \gg 1$) for all doublers $p = \pi_A$ the strong-coupling limit (static limit) exists and for the low-energy mode $p \sim 0$ the strong-coupling limit does not exist as discussed in section III, beside no hard spontaneous symmetry breaking occurs as discussed in eqs.(8,9) and ref. [17]. These crucial features are very different from the momentum-dependence of the Eichten-Preskill and

Smit-Swift models in their strong-coupling phases, where no such segment have been found. As the consequences of these features, our model has the following properties differentiated from the Eichten-Preskill and Smit-Swift models.

In the strong coupling phase (PMS) of the Smit-Swift model, only neutral sector of vector-like fermion exists in the spectrum which is computed by the strong-coupling expansion [30] method analogous to the one adopted in this paper. The wave-function renormalization of the composite neutral field is given by the vacuum expectation value of scalar fields $V(x)$, $z^2 = \langle V^\dagger(x)V(x+a) \rangle$ [30], which does not vanish as $p \rightarrow 0$, showing the neutral scalar-fermion bound state can not be excluded from the low-energy spectrum. However, as clearly demonstrated in section V, the neutral three-fermion-state, which dissolves into the three-fermion-cut, is excluded from the low-energy spectrum for its wave-function renormalization goes to zero as $p \sim 0$ and effective multifermion coupling $g_2(p)$ vanishes without any hard spontaneous symmetry breaking. Although the scalar-fermion coupling in the Smit-Swift model can be reduced so that the scalar-fermion bound state could disappear, the model runs into the phase of hard spontaneous symmetry breaking ($v \sim O(\frac{1}{a})$).

In the strong coupling phase of the Eichten-Preskill model, not only neutral sector but also charged sector of vector-like fermions exists in the spectra, both are computed by the strong-coupling expansion [15,19] method that is the exactly same as adopted in this paper. However, the strong coupling limit (the static limit) of the Eichten-Preskill model is regular at $p \sim 0$ differentiated from the singularity of the static limit at $p \sim 0$ of our model. This implies that the multifermion couplings as function of the momentum p of the Eichten-Preskill model are not appropriate so as to find a scaling region, these massive charged fermions remain as vector-like only for $p \sim \pi_A$ modes and not for the low-energy mode ($p \sim \pi_A$). Beside, the multifermion couplings as function of the momentum p of the Eichten-Preskill model develop a hard spontaneous symmetry breaking in an expected scaling region [19]. The multifermion couplings as function of the momentum p of our model are carefully made to avoid the hard spontaneous symmetry breaking in the possible scaling region \mathcal{A} , the shift-symmetry (5) plays extremely important role in this matter. The Eichten-Preskill model for $SU(5)$ and $SO(10)$ would have a chance to work, if its multifermion couplings are more carefully defined.

VIII. SOME REMARKS

In this paper, we have discussed the features of the spectrum of neutral and charged sectors appearing in segment \mathcal{A} (4). It is interesting to point out that in this scenario, doublers ($p \sim \pi_A$) are decoupled by a gauge-invariant mass term, while the low-energy modes with the “wrong” chirality are “decoupled” by the vanishing of their generalized form factor. However, it is still far from a definitive demonstration that chiral gauge theories in the low-energy limit can be achieved in this way, we need to have numerical simulations to show that this scenario is indeed realized in segment \mathcal{A} .

The whole spectrum in segment \mathcal{A} is gauge symmetric, and Ward identities of gauge symmetry are preserved. We can straightforwardly turn on the perturbative gauge interaction. By the strong coupling expansion in powers of $\frac{1}{g_2}$, we compute [18] three-point vertex function $\langle \Psi_R^i(x) \bar{\Psi}_R^j(y) A_\mu(z) \rangle$ and obtain the vertex of the $SU_L(2)$ gauge field coupling to the charged three-fermion-state (the leading order of gauge coupling $O(g)$),

$$\Lambda_{\mu RR}^{(1)}(p, p') = ig \frac{\tau^a}{2} \gamma_\mu P_R \cos \frac{(p+p')}{2}, \quad (68)$$

where the momenta of three-fermion-states $p, p' \sim \pi_A$. According the renormalization (27) of truncated Green functions with two three-fermion-operator insertions, the vertex of gauge coupling to the three-fermion-states is given by (Fig.5),

$$\Lambda_{\mu RR}^{(1)}(p, p') = ig \frac{\tau^a}{2} \gamma_\mu P_R \cos \frac{(p+p')}{2} Z_R(p) Z_R(p'). \quad (69)$$

For $p \sim \pi_A$ and $p' \sim \pi_A$, Z_R 's are positive definite constants, we thus renormalize the wave functions with respect to each doubler (25), as discussed in section IV. As a result, gauge coupling vertex (69) turns into (68). Although eq.(69) is obtained for $p, p' \sim \pi_A$ and $a^2 g_2 \gg 1$, it can be analytically continued to $p, p' \sim 0$. We find that in the limit of $p \rightarrow 0$ and $p' \rightarrow 0$, the coupling vertex (69) of three-fermion-operators and gauge boson vanishes as $O(p^8)$. This consistently corresponds to the dissolving of three-fermion-states into three-fermion-cuts in the low-energy limit. Since the propagator of charged three-fermion-states *positively* vanishes, there are no ghost states with negative norm coupling to gauge field through the Ward identity stemming from the gauge symmetry in segment \mathcal{A} .

The model presented in this paper cannot be considered as a realistic model reflecting all aspects of chiral gauge theories. We need to completely understand the relationship between the anomaly-free condition and the realization

of such dynamics discussed in the paper. We also need to understand what kind of 't Hooft condition for anomaly matching [35], fermion-number violation and Witten's $SU(2)$ global anomaly [36] would play in such dynamics.

In this approach, the left-handed field $\psi_L^i(x)$ is the doublet of the $SU_L(2)$ chiral gauge group. However, it can be generalized to be the left-handed field (complex representations) of any anomaly-free chiral gauge group (e.g. $SU(5)$ and $SO(10)$). A right-handed field (spectator) χ_R , that is a singlet of the chiral gauge group, can be introduced and coupled to the left-handed field in the same way as the multifermion couplings given in (1). As for the right-handed field ψ_R^i of chiral gauge groups, we can analogously introduce a left-handed spectator field χ_L that is a singlet of the chiral gauge groups, and couple it to the right-handed field ψ_R^i in the same way as the multifermion couplings in the action (1) with $L \leftrightarrow R$. This indicates that such a formulation of chiral gauge theories is actually quite general.

To be more specific, we take the anomaly-free chiral gauge group of the Standard Model as an example. In this realistic case, $\psi_L^i(x)$ can be both left-handed lepton doublets and left-handed quark doublets. The candidate for a right-handed spectator field χ_R could be the right-handed neutrino ν_R . As for the right-handed fields ψ_R^i with respect to $U_Y(1)$, we can introduce an additional left-handed spectator field χ_L (a $SU_L(2) \otimes U_Y(1)$ singlet) coupling to the right-handed fields ψ_R^i as that in eq.(1). These spectator fields ν_R and χ_L are free and decouple from other particles due to the ν_R - and χ_L -shift-symmetry acting on them. In this way, we can in principle have a gauge-invariant formulation of the Standard Model on the lattice. In practice, non-perturbative analysis, which can be done analytically to show whether such a formulation gives the low-energy Standard Model, should be more or less similar to (certainly more complicated than) that discussed in this paper and references [17,18]. However, the spectator field χ_L might not be necessary. Alternatively, the 't Hooft vertex in the lattice formulation of the Standard Model [14] provides a scenario in which fermion-number conservation is explicitly violated, and all chiral fermions find their partners with opposite chirality within the Standard Model. The formulation of a realistic Standard Model with multifermion couplings and all features of dynamics discussed in this paper are an extremely interesting subject for future studies.

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Appendix A

The propagator of ψ_L^i is given by eq.(14) ($p \sim 0$),

$$S_{LL}^{ij}(p) = \delta_{ij} P_L \hat{p} P_R, \quad \hat{p} = \frac{\frac{i}{a} \sum_{\mu} \gamma^{\mu} \sin p_{\mu}}{\frac{1}{a^2} \sum_{\mu} \sin^2 p_{\mu} + M^2(p)}. \quad (70)$$

The Feynman diagram (see Fig. 6) is given by,

$$\begin{aligned} \hat{\sigma}^{ij}(p) &= \delta^{ij} P_R \sigma(p) P_L \\ \sigma(p) &= -N_f \int_{qk} \lambda S_{RR}(p+q) \text{tr} \left[S_{RR}(k - \frac{q}{2}) S_{LL}(k + \frac{q}{2}) \right] \\ &= -N_f \int_{k,q} \lambda S_{RR}(p+q) R(k, q), \end{aligned} \quad (71)$$

where S_{LL}, S_{RR} are (14,15) and $R(k, q)$ is eq. (39) and

$$\lambda = \left(4g_2 w(p-k) w(k + \frac{q}{2}) \right)^2 = \frac{1}{N_f^2} a^{-4} \left(4\tilde{g}_2 w(p-k) w(k + \frac{q}{2}) \right)^2. \quad (72)$$

The wave-function renormalization $Z_2(p)$ of $\psi_L^i(x)$ in eq. (57) can be calculated by using the train approximation (see Fig. 4),

$$\begin{aligned} Z_2^{-1} S_{LL}^{ij}(p) &= P_L (\hat{p} + \hat{p} \sigma \hat{p} + \hat{p} \sigma \hat{p} \sigma \hat{p} + \dots) P_R \delta^{ij} \\ &= S_{LL}^{ij}(p) \left(\frac{1}{1 - \sigma \hat{p}} \right), \end{aligned} \quad (73)$$

and one gets

$$Z_2 = 1 - \sigma \hat{p}. \quad (74)$$

With eq. (71) one can get

$$\sigma\hat{p} = -\frac{1}{N_f} \int_{k,q} \left(4\tilde{g}_2 w(p-k)w(k+\frac{q}{2}) \right)^2 S_{RR}(p+q)S_{LL}(p)R(k,q). \quad (75)$$

By substituting eq. (75) into (74), one gets eq. (58).

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APPENDIX: FIGURE CAPTIONS

- Figure 1:** The two-point Green functions of composite three-fermion-operators at strong coupling $g_2 \gg 1$ for $p \sim \pi_A$, indicating three-fermion-states.
- Figure 2:** 1PI truncated Green functions of an elementary field and one three-fermion-operator insertion, i.e., the generalized form factors of three-fermion-states.
- Figure 3:** The two-point Green functions of composite three-fermion-operators in the effective weak coupling for $p \sim 0$, indicating three-fermion-cuts.
- Figure 4:** The train approximation to the propagators of the elementary fields ψ_L^i and χ_R .
- Figure 5:** Gauge boson coupling to three-fermion-states.
- Figure 6:** A single bubble diagram in the weak-coupling expansion (see Figs.3,4).

FIG. 1. Fig.1, Fig.2 and Fig.3

FIG. 2. Fig.5, Fig.6 and Fig.7